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The generation of sound by vorticity waves in swirling duct flows

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Swirling flow in an axisymmetric duct can support vorticity waves propagating parallel to the axis of the duct. When the cross-sectional area of the duct changes a portion of the wave energy is scattered into secondary vorticity and sound waves. Thus the swirling flow in the jet pipe of an aeroengine provides a mechanism whereby disturbances produced by unsteady combustion or turbine blading can be propagated along the pipe and subsequently scattered into aerodynamic sound. In this paper a linearized model of this process is examined for low Mach number swirling flow in a duct of infinite extent. It is shown that the amplitude of the scattered acoustic pressure waves is proportional to the product of the characteristic swirl velocity and the perturbation velocity of the vorticity wave. The sound produced in this way may therefore be of more significance than that generated by vorticity fluctuations in the absence of swirl, for which the acoustic pressure is proportional to the square of the perturbation velocity. The results of the analysis are discussed in relation to the problem of excess jet noise.

1. Introduction

The noise produced by a low subsonic jet is dominated by sources associated with regions of the flow within or in the immediate vicinity of the jet pipe (Ffowcs Williams 1977), and is usually referred to as *excess jet noise* to distinguish it from the noise arising from the turbulent mixing of the jet with the ambient atmosphere. The sound pressure level of the excess noise scales typically on a power of the jet velocity U lying between 3 and 6, whereas pure jet mixing noise varies according to Lighthill's (1952) U^8 law. Experiments (Hoch & Hawkins 1973) indicate that nozzle-based sources tend to be important in determining the overall structure of the acoustic field-shape only at the lower jet velocities. However, there is an increasing body of both theoretical and experimental evidence (Ffowcs Williams 1977; Crighton 1975*a*) which suggests that at the higher jet velocities excess noise sources provide a significant contribution in radiation directions forward of the aircraft and at 90° to the flight path, and in particular are affected by forward motion of the aircraft in a manner which largely accounts for the so-called forward-arc amplification of the sound with increasing flight velocity.

Several possible mechanisms of excess jet noise have been examined in terms of detailed analytical models. Candel (1972), Marble (1973), Cumpsty & Marble (1974), Ffowcs Williams & Howe (1975) and Howe (1975) have discussed the manner in which

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temperature inhomogeneities generate sound when convected in a non-uniform mean flow. Cumpsty & Marble (1974) and Howe (1975) have investigated analogous problems in which the convected inhomogeneity is composed of a localized field of vorticity. The sound produced when such aerodynamic sources interact with struts and sharp trailing edges has been analysed by Jones (1972), Crighton (1975*a*) and Howe (1976, 1977). Unsteady combustion processes generate pressure fluctuations which propagate down the jet pipe, and a linearized model of their subsequent interaction with the exterior jet flow prior to radiation into the far field has been proposed by Munt (1976).

The study of excess jet noise is also of great interest in connexion with large-scale structures in the free jet. Sound radiated along the jet pipe from such excess jet noise mechanisms provides a possible stimulation of the large-scale instabilities of the free jet (Bishop, Ffowcs Williams & Smith 1971; Liu 1974; Merkine & Liu 1975) and could thus manifest itself via the radiation properties of the stimulated large-scale structure in addition to via direct radiation into the far field (Crighton 1975*b*). However, this would form the subject for study elsewhere and will not be discussed further in this paper.

The flow in all real engine ducts is swirling to some extent because of the presence of a mean axial component of vorticity induced by turbines and guidevanes. Schwartz (1973) has argued that an increase in the degree of swirl could lead to a reduction in the intensity of turbulent fluctuations because of the action of centrifugal forces and this, in turn, should result in a reduced level of the jet mixing noise. Data obtained by Schwartz (1973) and Whitfield (1975) suggest that the only noise benefits are likely to be in the rear arc, i.e. in directions of propagation downstream of the jet nozzle, and then only at rather low frequencies. Whitfield's (1975) data indicate that such frequencies are below those of aeronautical interest. The effect of swirl on the turbulence in a free jet remains, however, to be examined theoretically and understood.

In this paper a model problem is examined which indicates that the presence of swirl introduces an additional excess noise mechanism, and thereby increases the efficiency with which disturbances in the upstream region of the jet pipe are transformed into acoustic waves which radiate from the nozzle exit. This mechanism depends on the possibility that under appropriate conditions essentially incompressible dispersive vorticity waves can propagate along the duct parallel to the swirl axis. Turbulence, unsteady combustion and blading processes may be expected to be responsible for the generation of these waves. They are, of course, acoustically silent when propagating in a duct of uniform cross-section, but variations in duct geometry and the nozzle exit would both be expected to act as scattering centres at which secondary vorticity and acoustic waves are formed.

We consider the problem of the generation of sound which occurs when such a vorticity wave encounters a contraction (figure 1*a*) or a necking (figure 1*b*) in a duct of infinite extent. This may be taken to model the situation in which the sound is generated sufficiently far upstream of the nozzle exit for the analysis of its subsequent radiation into free space to be treated separately. Ffowcs Williams & Howe (1975) have shown that at small mean flow Mach numbers a reliable estimate of the free-space radiation may be obtained by applying Rayleigh's (1945, chap. 16) transfer operator to the sound wave when it reaches the nozzle exit, a procedure which is also

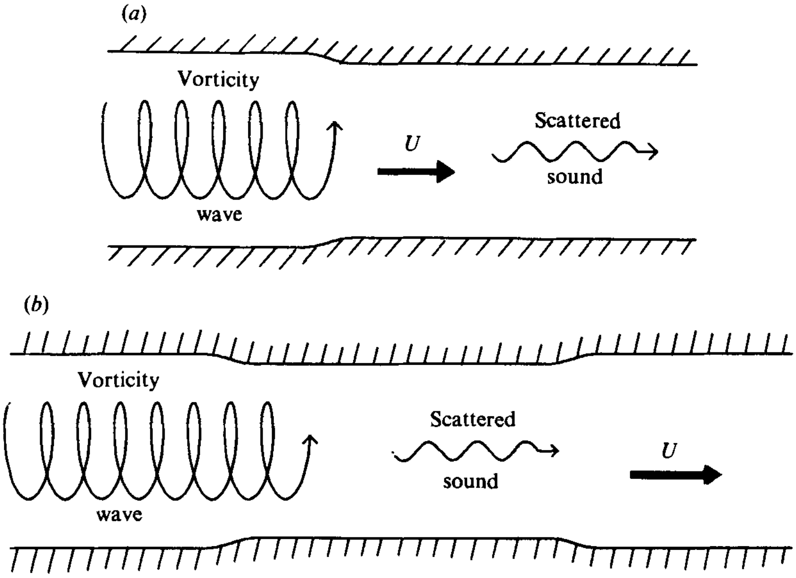

FIGURE 1. A duct of infinite extent contains a steady, low Mach number swirling flow. A vorticity wave supported by the swirling flow is incident from upstream on (*a*) a small contraction and (*b*) a necking (of length $2L$) in the cross-sectional area of the duct where secondary vorticity and sound waves are produced.

expected to be of value when the scattering centre is located within an acoustic wavelength of the nozzle exit.

The scattering problem is amenable to straightforward analysis only when it is assumed that the proportional change in the cross-sectional area of the duct is small. The characteristics of the emitted sound may then be determined directly from the equations of aerodynamic sound theory provided that the mean-flow Mach number and the dominant acoustic frequency are small enough that the disturbed flow in the scattering region may be regarded as essentially incompressible. This approach is very much analogous to that which was adopted by Lighthill (1953) in considering the interaction of sound with turbulence, and which was subsequently extended by Curle (1955) to problems involving solid boundaries.

The aerodynamic sound problem is formulated and solved in § 2 using a formula given by Howe (1975) describing the generation of sound by turbulence in non-uniform duct flows. Howe was concerned with localized turbulent eddies convected in an otherwise irrotational mean flow, and showed that the amplitude of the radiated pressure waves was proportional to the square of the turbulent fluctuation velocity. The mean flow had an appreciable effect on the frequency of the sound, but not on its intensity. This is not the case, however, when the duct flow is swirling and possesses a mean component of vorticity. In the leading approximation the amplitude of the sound is arguably much larger, being proportional to the product of the characteristic swirl velocity and the perturbation velocity associated with the vorticity wave. In § 3 the case of a harmonic vorticity wave is examined in detail for an arbitrary swirl velocity profile, and leads to a general expression for the scattered sound. This is specialized in § 4 to duct flow in solid-body rotation and the examples treated here are discussed (§ 5) in relation to the problem of excess jet noise.

2. The aerodynamic sound problem

Consider a steady flow in a nominally uniform duct of circular cross-section. Take cylindrical polar co-ordinates (x, r, θ) of which x is measured along the axis of symmetry of the duct and in terms of which the mean flow is given by

$$\mathbf{U} = (U, 0, w_0(r)). \tag{2.1}$$

The axial velocity U is constant, but the swirl velocity $w_0(r)$ is a function of the radial co-ordinate r.

If the flow encounters a change in the cross-sectional area of the duct, such as the contraction of figure 1 (*a*) or the necking of figure 1 (*b*), the motion downstream of the area change eventually returns to one of steady cylindrical flow unless the characteristic swirl velocity is large compared with U (Batchelor 1967, p. 550). In the latter case it is possible for the change in cross-section to produce an axisymmetric train of steady vorticity waves in its wake. If ϵ is a small parameter which characterizes the proportional change in the radius of the duct, it follows that the change in the mean flow is also of order ϵ and may be regarded analytically as a steady field generated by the scattering of the incident flow at the area change. In aeronautical applications the swirl velocity is relatively small and we shall therefore discount the possibility of there being a steady train of waves downstream of the area change. In general the swirling flow can support unsteady vorticity waves which in the case of an engine jet pipe could develop in regions of unsteady combustion and turbine blading. They will also be scattered at the area change, where secondary vorticity and acoustic waves will be generated.

The acoustic component of the scattered field may be determined directly from the Lighthill (1952) acoustic-analogy theory of aerodynamic sound. For non-uniform duct flows it is convenient to recast the Lighthill theory into a form in which the stagnation enthalpy

$$B = h + \tfrac{1}{2}\mathbf{v}^2 \tag{2.2}$$

assumes the role of the fundamental acoustic variable. In this expression h is the specific enthalpy of the fluid and $\mathbf{v}$ the velocity. In the case of an ideal gas, and in situations in which it is permissible to neglect dissipation and transport phenomena, B is determined in terms of the vorticity $\boldsymbol{\omega}$ and the specific entropy S by means of the inhomogeneous wave equation

$$\left\{\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D}{Dt}\right) + \frac{1}{c^2}\frac{D\mathbf{v}}{Dt}\cdot\nabla - \nabla^2\right\}B = \operatorname{div}\{\boldsymbol{\omega}\wedge\mathbf{v} - T\nabla S\} - \frac{1}{c^2}\frac{D\mathbf{v}}{Dt}\cdot\{\boldsymbol{\omega}\wedge\mathbf{v} - T\nabla S\} \tag{2.3}$$

(Howe 1975), where c is the speed of sound, T the temperature and D/Dt the material derivative. In the present discussion it will be assumed further that the flow is of uniform temperature and entropy. In the absence of vorticity waves in the duct the stagnation enthalpy B_0, say, is constant on the Bernoulli surfaces formed by intersecting families of streamlines and vortex lines. Accordingly only the time-dependent fluctuations in the source terms need be retained on the right of (2.3), and B may be reinterpreted as the local perturbation stagnation enthalpy in the manner described in detail by Howe (1976).

In order to solve (2.3) we introduce the following important approximations. First the mean-flow velocity is taken to be sufficiently small that $M^2 \ll 1$, where $M = U/c$. The sound speed c may then be regarded as constant throughout the flow, and terms of relative order M^2 maybe neglected in (2.3). In aeronautical applications the dominant Strouhal number $St \equiv 2fa/U \lesssim 1$, where a is the duct radius and f (Hz) is the frequency of the generated sound (Lush 1971). At small values of the Mach number we are thus concerned with situations in which the characteristic wavelength of the sound is large compared with the diameter of the duct, and we shall suppose M to be sufficiently small that the dimension of the region in which the area change occurs is small on the scale of the wavelength. It then follows that the propagation of the incident vorticity wave may be described by the equations of incompressible flow. Further, for M less than about 0·5 the above restriction on the Strouhal number implies that the frequencies of interest are below the cut-off frequencies of all non-axisymmetric acoustic modes, and that upstream and downstream of the area change the scattered sound consists of *plane* waves.

The general equation (2.3) may now be approximated by

$$\left\{\frac{1}{c^2}\left(\frac{\partial}{\partial t}+\mathbf{U}.\frac{\partial}{\partial \mathbf{x}}\right)^2-\nabla^2\right\}B = \operatorname{div}(\boldsymbol{\omega}\wedge\mathbf{v}). \tag{2.4}$$

Since the scattered acoustic waves are one-dimensional in the distant field, the fluctuations in B are related to the acoustic pressure perturbation p and the mean density ρ_0 as $x \to +\infty$ by

$$B \simeq (1+M)\,p/\rho_0, \tag{2.5}$$

a result which follows from (2.2) on noting that when the entropy is uniform $h = \int dp/\rho$, where ρ is the local fluid density. In (2.5), $M = U/c$ is the Mach number of the axial mean flow introduced above, which, since $\epsilon \ll 1$, takes approximately equal values on either side of the area change.

The formal solution of (2.4) has been obtained by Howe (1975), who shows that the plane acoustic pressure waves radiated downstream of the area change may be expressed in the form

$$p \simeq \frac{1}{2A(1+M)^2}\int [\rho_0 u_i u_j e_{ij}]\, d^3\mathbf{y}, \tag{2.6}$$

where A is the uniform cross-sectional area of the duct, $\rho_0 u_i u_j$ the *fluctuating* component of the Reynolds stress, and the integration is performed over the volume of the duct. The quantity in square brackets is evaluated at the retarded time

$$t - x/[c(1+M)],$$

x being the position of an observer located far downstream (the origin of co-ordinates being in the vicinity of the area change). The second-order tensor $\mathbf{e}$ is determined by the geometry of the duct. It is given by

$$e_{ij} = \frac{\partial^2\phi^*}{\partial x_i\,\partial x_j} \equiv \frac{1}{2}\left(\frac{\partial U_i^*}{\partial x_j}+\frac{\partial U_j^*}{\partial x_i}\right) \tag{2.7}$$

and is the rate-of-strain tensor associated with steady, incompressible, irrotational flow at velocity $\mathbf{U}^*$ through the duct, the potential $\phi^*(\mathbf{x})$ being normalized such that $\partial\phi^*/\partial x \to 1$ as $x \to +\infty$. The level of the radiated sound is thus proportional to the rate of working of the Reynolds stress in the rate-of-strain field of the duct.

3. The scattered sound: the case of a harmonic vorticity wave

Let (u, v, w) denote respectively the (x, r, θ) components of the flow velocity. Since $\phi^*(\mathbf{x})$ necessarily describes an axisymmetric duct flow, it follows that

$$u_i u_j e_{ij} = u^2 e_{xx} + 2uve_{xr} + v^2 e_{rr} + w^2 e_{\theta\theta} \tag{3.1}$$

(see, for example, Batchelor 1967, p. 602). Only the time-dependent part of this is required, and in a first approximation we may set

$$u = U + u', \quad v = v', \quad w = w_0(r) + w', \tag{3.2}$$

where the primed quantities denote the small perturbations associated with an incident vorticity wave. The fluctuating part of (3.1) reduces to

$$u_i u_j e_{ij} \simeq 2U(u'e_{xx} + v'e_{xr}) + 2w_0 w' e_{\theta\theta} \tag{3.3}$$

when second-order terms are discarded.

Again, axisymmetry implies that e_{ij} does not depend on the azimuthal co-ordinate θ. This means that in the present linear theory only axisymmetric disturbances (u', v', w') contribute to the radiated sound, the remaining components of the disturbed flow integrating to zero in (2.6). The analysis may now be considerably simplified because the axisymmetric part of the flow can be expressed in terms of a Stokes stream function ψ, with

$$u' = r^{-1}\partial\psi/\partial r, \quad v' = -r^{-1}\partial\psi/\partial r. \tag{3.4}$$

The corresponding formula for the azimuthal velocity component w' will be considered below.

We are now in a position to examine the contributions of the various terms in (3.3) to the radiation integral (2.6). Consider the first term on the right of (3.3). Since

$$e_{xx} = \partial^2\phi^*/\partial x^2, \quad e_{xr} = \partial^2\phi^*/\partial x\,\partial r, \tag{3.5}$$

the corresponding contribution to (2.6) may be expressed in the form

$$\begin{aligned} p &\simeq \frac{\rho_0 U}{A(1+M)^2}\int\left[\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial^2\phi^*}{\partial y^2} - \frac{1}{r}\frac{\partial\psi}{\partial y}\frac{\partial^2\phi^*}{\partial r\,\partial y}\right] r\,d\theta\,dr\,dy \\ &= \frac{2\pi\rho_0 U}{A(1+M)^2}\int_{-\infty}^{\infty} dy \int_0^a \left[\frac{\partial}{\partial r}\left(\psi\frac{\partial^2\phi^*}{\partial y^2}\right)\right] dr \\ &\quad - \frac{2\pi\rho_0 U}{A(1+M)^2}\int_{-\infty}^{\infty} dy \int_0^a \left[\frac{\partial}{\partial y}\left(\psi\frac{\partial^2\phi^*}{\partial r\,\partial y}\right)\right] dr, \end{aligned} \tag{3.6}$$

where in the linearized approximation the radius of the duct is taken to be constant and equal to a, that in the upstream flow region. Each of these integrals may be shown to vanish by integration by parts. The first because the perturbation stream function must vanish at the centre and at the wall of the duct, and the second because $\partial^2\phi^*/\partial r\,\partial y$ tends rapidly to zero as $y \to \pm\infty$. Thus no sound is produced by the first term on the right of (3.3). The acoustic response is therefore determined by the second part of (3.3), which depends on the swirl component $w_0(r)$ of the mean velocity. Thus, noting that $e_{\theta\theta} = r^{-1}\partial\phi^*/\partial r$, we have

$$p \simeq \frac{2\pi\rho_0}{A(1+M)^2}\int_{-\infty}^{\infty} dy \int_0^a dr \left[w_0(r)\, w' \frac{\partial\phi^*}{\partial r}\right]. \tag{3.7}$$

In order to proceed beyond this formal result it is necessary to introduce a specific form for the disturbance velocity w'. We are assuming that the incident vorticity wave may be described by the equations of incompressible flow theory, and have shown that only the axisymmetric component of w' is required. Further, in the linear approximation variations in the duct geometry may be neglected in determining the form of the incident wave, and the three components of the momentum equation reduce to

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}, \tag{3.8a}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)v' - \frac{2w_0(r)\,w'}{r} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial r}, \tag{3.8b}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)w' + \frac{w_0(r)\,v'}{r} + v'\frac{\partial w_0}{\partial r} = 0, \tag{3.8c}$$

where p' is the incompressible pressure perturbation. In solving these equations it may be assumed that the duct has constant radius a.

When the pressure p' is eliminated and the Stokes stream function (3.4) is introduced, the system (3.8) becomes

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\left\{\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial x^2}\right\}+\frac{1}{r^3}\frac{\partial}{\partial r}((w_0 r)^2)\frac{\partial^2\psi}{\partial x^2} = 0. \tag{3.9}$$

An elementary solution of this equation which represents an axisymmetric disturbance in a duct of uniform cross-sectional area is

$$\psi = \Psi(r)\exp\{i(kx-\omega t)\}, \tag{3.10}$$

where the wavenumber k is determined in terms of the radian frequency ω through the boundary condition which requires that ψ should vanish at $r = a$ (see § 4 below). A general axisymmetric disturbance may be constructed by suitably superposing waves of this type. On substituting (3.10) into (3.9) we find that $\Psi(r)$ satisfies

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\right)+\left\{\frac{1}{(\omega-Uk)^2r^3}\frac{\partial}{\partial r}((w_0 r)^2)-1\right\}k^2\Psi = 0, \tag{3.11}$$

and by making use of (3.8c) we obtain the following representation for the azimuthal component w' of the disturbance velocity:

$$w' = \frac{-k\Psi(r)}{(\omega-Uk)\,r^2}\frac{\partial}{\partial r}(w_0\, r)\exp\{i(kx-\omega t)\}. \tag{3.12}$$

This result may be inserted into (3.7) to give the following expression for the pressure wave radiated downstream:

$$p \simeq \frac{-\pi\rho_0 k}{A(1+M)^2(\omega-Uk)}\int_{-\infty}^{\infty}dy\int_0^a dr\frac{\Psi(r)}{r^3}\frac{\partial}{\partial r}((w_0 r)^2)\,\frac{\partial\phi^*}{\partial r}\exp\{i(ky-\omega[t])\}, \tag{3.13}$$

where $[t] = t - x/c(1+M)$.

Now $\phi^* \equiv \phi^*(r,y)$ describes a hypothetical axisymmetric irrotational flow in the linearly non-uniform duct. Define

$$\Phi^*(r,\kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi^*(r,y)\,e^{i\kappa y}\,dy, \tag{3.14}$$

the Fourier transform of ϕ^* with respect to the axial co-ordinate y. Then Φ^* satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi^*}{\partial r}\right) - \kappa^2 \Phi^* = 0, \tag{3.15}$$

the Fourier transform of the axisymmetric Laplace equation, and (3.13) becomes

$$p \simeq \frac{-2\pi^2 \rho_0 k e^{-i\omega[t]}}{A(1+M)^2(\omega - Uk)} \int_0^a \xi(r,k) \frac{\Psi(r)}{r^3} \frac{\partial}{\partial r}((w_0 r)^2)\, dr, \tag{3.16}$$

where $$\xi(r,k) = \partial \Phi^*(r,k)/\partial r.$$

The remaining integral in (3.16) may be evaluated by first noting that (3.15) implies that ξ satisfies

$$\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\xi)\right\} - k^2 \xi = 0. \tag{3.17}$$

Multiply this by $\Psi(r)$ and (3.11) by ξ, subtract and integrate from $r = 0$ to a to obtain

$$\frac{k^2}{(\omega - Uk)^2} \int_0^a \frac{\xi \Psi}{r^3} \frac{\partial}{\partial r}((w_0 r)^2)\, dr = \int_0^a \left\{ \Psi \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\xi)\right) - \xi r \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right)\right\} dr. \tag{3.18}$$

The integral on the right can be evaluated by integrating by parts, and on noting that Ψ must vanish at $r = 0$ and $r = a$ it follows readily that (3.16) becomes

$$p \simeq \frac{2\pi^2 \rho_0 (\omega - Uk)\, e^{-i\omega[t]}}{A(1+M)^2 k} \left(\frac{\partial \Psi}{\partial r} \frac{\partial \Phi^*}{\partial r}(r,k)\right)_{r=a}. \tag{3.19}$$

This result expresses the radiation in terms of the boundary values of $\partial\Psi/\partial r$ and $\partial\Phi^*(k)/\partial r$, and indicates that it should be possible to express the scattered sound explicitly in terms of the shape of the duct.

Let the boundary of the duct be expressed in the form

$$r = a + \epsilon f(x), \tag{3.20}$$

where the function $f(x)$ is $O(1)$. When ϵ vanishes we have $\phi^* = x + \text{constant}$. Thus for $\epsilon \neq 0$, $\partial\phi^*/\partial x$ must depart from unity by an amount proportional to ϵ, and it follows that correct to $O(\epsilon)$ the condition that the normal irrotational velocity $\partial\phi^*/\partial n$ vanishes on the surface of the duct is

$$\left(\frac{\partial \phi^*}{\partial r}\right)_{r=a} = \epsilon \frac{\partial f}{\partial x}. \tag{3.21}$$

Hence (3.19) becomes

$$p \simeq \frac{2\pi^2 \epsilon \rho_0 (\omega - Uk)}{A(1+M)^2 k} \mathscr{F}(k) \left(\frac{\partial \Psi}{\partial r}\right)_{r=a} \exp\left\{-i\omega\left(t - \frac{x}{c(1+M)}\right)\right\}, \tag{3.22}$$

where $$\mathscr{F}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} e^{ikx}\, dx \tag{3.23}$$

is the Fourier transform of the axial slope of the duct wall.

Equation (3.22) is the principal result of this section, and determines the scattered sound in terms of the shape of the duct boundary and the axial component at the boundary of the perturbation velocity associated with the incident vorticity wave (3.10). No assumption has been made regarding the form of the swirl velocity profile $w_0(r)$. In the absence of swirl (3.22) vanishes identically, because in that case linear theory requires that a vortical disturbance convects at the mean-flow velocity U and hence that $\omega - Uk \equiv 0$. This is in accordance with the general result obtained by Howe (1975), who showed that for an irrotational mean flow the sound produced by vorticity located near an area change is of second order in the perturbation velocity.

4. Rigid-body rotational flow

Let us now consider the particular case in which the mean flow in the duct is one of solid-body rotation at angular velocity $\Omega > 0$ about the axis of symmetry. The swirl velocity profile is therefore given by

$$w_0(r) = \Omega r, \tag{4.1}$$

and equation (3.11) for the radial dependence Ψ of the incident vorticity wave $\Psi(r)\exp\{i(kx-\omega t)\}$ reduces to

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\right)+\left\{\frac{4\Omega^2}{(\omega-Uk)^2}-1\right\}k^2\Psi = 0. \tag{4.2}$$

The solution of this equation which remains finite at $r = 0$ is

$$\Psi(r) = \delta r J_1(\gamma r) \tag{4.3}$$

(cf. Batchelor 1967, p. 546), where δ is a parameter characterizing the amplitude of the wave and

$$\gamma^2 = \left\{\frac{4\Omega^2}{(\omega-Uk)^2}-1\right\}k^2. \tag{4.4}$$

The condition that the perturbation stream function must vanish on the wall $r = a$ of the duct implies that

$$J_1(\gamma a) = 0. \tag{4.5}$$

This characteristic equation determines the permissible values of γ, and (4.4) then furnishes the following dispersion relation between k and ω:

$$\omega = Uk \pm 2\Omega k/(\gamma^2+k^2)^{\frac{1}{2}}. \tag{4.6}$$

The zeros of $J_1(x)$ lie on the real axis (Watson 1966, p. 483), so that (4.6) defines ω as a real function of k, i.e. the vorticity wave is stable.

Wave energy propagates at the group velocity $\partial\omega/\partial k$, which is given by

$$\partial\omega/\partial k = U \pm 2\Omega\gamma^2/(\gamma^2+k^2)^{\frac{3}{2}}. \tag{4.7}$$

It is positive for either choice of sign in (4.6) provided that the maximum swirl velocity Ωa does not exceed $\frac{1}{2}U\gamma a$. This is certainly the case in aeronautical applications because $\Omega a/U$, the inverse Rossby number, is unlikely to be greater than about 0·5 and the minimum non-trivial value of γa obtained from (4.5) is 3·83. The waves corresponding to the plus/minus sign in (4.6) may therefore be designated 'fast'/'slow' waves in the present context, and fulfil the condition of our initial hypothesis that the vorticity waves may be regarded as generated by disturbances located upstream of the area change.

Next, it follows from (4.3) and the known properties of Bessel functions that

$$(\partial\Psi/\partial r)_{r=a} = \delta\gamma a J_0(\gamma a). \tag{4.8}$$

Hence, substituting this into the formula (3.22) for the downstream scattered sound we find that

$$p \simeq \pm \frac{4\pi^2 \epsilon \rho_0 \Omega \delta \gamma a}{A(1+M)^2 (\gamma^2+k^2)^{\frac{1}{2}}} J_0(\gamma a) \mathscr{F}(k) \exp\left\{-i\omega\left(t - \frac{x}{c(1+M)}\right)\right\}, \tag{4.9}$$

use having been been made of the dispersion relation (4.6).

This result shows that in order of magnitude $p \sim \rho_0 \Omega a v$, where v may be regarded as the root-mean-square perturbation velocity of the vorticity wave obtained by averaging over the cross-section of the duct and over a wavelength $2\pi/k$ of the disturbance. It is convenient to normalize p with respect to $\rho_0 U v$. Apart from a constant independent of k, v is given by the integral

$$v^2 = \frac{1}{a^2}\int_0^a \left\{\left(\frac{\partial \Psi}{\partial r}\right)^2 + \left(1 + \frac{4\Omega^2}{(\omega - Uk)^2}\right) k^2 \Psi^2\right\} \frac{dr}{r}. \tag{4.10}$$

Using in turn (4.6) and (4.3) we have

$$\begin{aligned} v^2 &= \frac{1}{a^2}\int_0^a \left\{\left(\frac{\partial \Psi}{\partial r}\right)^2 + (\gamma^2 + 2k^2)\Psi^2\right\}\frac{dr}{r} \\ &= \frac{\delta^2}{a^2}\int_0^a \{\gamma^2 J_0(\gamma r)^2 + (\gamma^2+2k^2) J_1(\gamma r)^2\} r\,dr. \end{aligned} \tag{4.11}$$

The integrand is an exact differential (McLachlan 1955, p. 103), and recalling that $J_1(\gamma a) = 0$, we obtain

$$v = \delta(\gamma^2 + k^2)^{\frac{1}{2}} J_0(\gamma a). \tag{4.12}$$

Hence (4.9) may now be expressed in the form

$$\frac{p}{\rho_0 U v} \simeq \pm \frac{4\pi\epsilon(\Omega a/U)\mathscr{F}(k)}{(1+M)^2[(\gamma a)^2 + (ka)^2]} \exp\left\{-i\omega\left(t - \frac{x}{c(1+M)}\right)\right\}, \tag{4.13}$$

where it has been noted that $A = \pi a^2$.

This expression illustrates the manner in which the acoustic response to the incident vorticity wave rapidly diminishes with increasing values of ka, the ratio of the duct radius to the hydrodynamic wavelength. If l is representative of the axial distance over which the area change occurs, i.e. for which $\partial f/\partial x$ is significantly different from zero, the Fourier transform $\mathscr{F}(k)$ is significant only for $kl \lesssim 1$. In practice l is likely to be much smaller than the radius a of the duct, in which case (4.13) implies that the principal contribution to the scattered sound is provided by vorticity disturbances whose wavelength greatly exceeds l, i.e. for which kl is small. When this is so a negligible error will be incurred by taking $l = 0$.

Thus in the case of the simple contraction of figure 1 (a) we shall suppose that the change in area takes place at $x = 0$, and that equation (3.20) for the duct boundary is

$$r = a - a\epsilon H(x), \tag{4.14}$$

$H(x)$ being the Heaviside step function. This means that $\mathscr{F}(k) = -a/2\pi$, and that (4.13) becomes

$$\frac{p}{\rho_0 U v} \simeq \pm \frac{2\epsilon(\Omega a/U)\gamma a}{(1+M)^2[(\gamma a)^2 + (ka)^2]} \exp\left\{-i\omega\left(t - \frac{x}{c(1+M)}\right)\right\}. \tag{4.15}$$

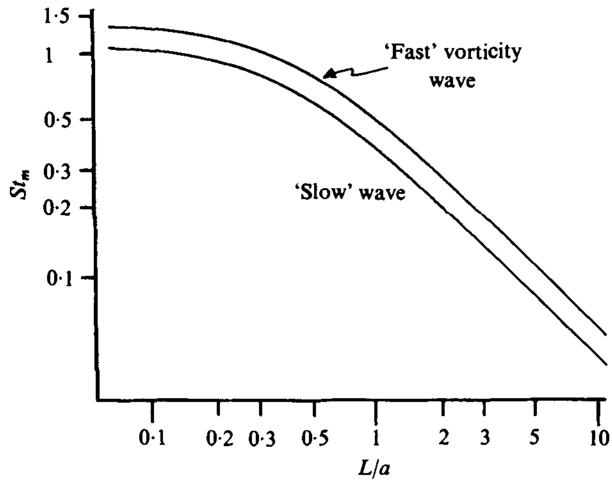

FIGURE 2. Variation of the peak Strouhal number St_m of the interaction of a vorticity wave with a necking of length $2L$ in a duct of mean radius a. $\Omega a/U = 0{\cdot}3$.

For the situation depicted in figure 1 (*b*), we assume that the necking extends from $x = -L$ to $x = L$ and that

$$r = a - a\epsilon\{H(x+L) - H(x-L)\}. \tag{4.16}$$

Hence $\mathscr{F}(k) = (ia/\pi)\sin kL$, and the expression (4.13) for the scattered pressure wave is

$$\frac{p}{\rho_0 Uv} \simeq \pm \frac{4i\epsilon(\Omega a/U)\,\gamma a \sin kL}{(1+M)^2[(\gamma a)^2 + (ka)^2]} \exp\left\{-i\omega\left(t - \frac{x}{c(1+M)}\right)\right\}. \tag{4.17}$$

This indicates that very little acoustic energy is scattered from vorticity waves whose wavelength is large compared with the length of the contraction in the duct.

5. Application to excess jet noise

The dimensionless parameter γa which appears in (4.15) and (4.17) may be taken to be any one of the zeros of $J_1(\gamma a)$ other than $\gamma a = 0$. For increasingly large values of γa, corresponding to vorticity waves possessing a progressively more complicated transverse structure, (4.15) and (4.17) indicate that the efficiency with which sound is generated decreases.

Equation (4.15) reveals that the sound produced by a contraction in the cross-sectional area of the duct is dominated by vorticity waves whose hydrodynamic wavelengths are large compared with the duct radius. On the other hand, when the contraction is of finite length $2L$ the strength of the radiation field given by (4.17) is controlled by the Fourier components of the incident disturbance having a small but finite value of the wavenumber k. In this case the radian frequency ω for which the acoustic response is a maximum depends on the ratio of the length of the contraction to the diameter of the duct, i.e. on L/a. This dominant frequency may be expressed as a Strouhal number $St_m = 2fa/U$, where $2\pi f = \omega$, and its variation with L/a is illustrated in figure 2 for the lowest-order radial mode ($\gamma a = 3{\cdot}83$) and a swirl of $\Omega a/U = 0{\cdot}3$. The upper and lower curves in the figure correspond respectively to

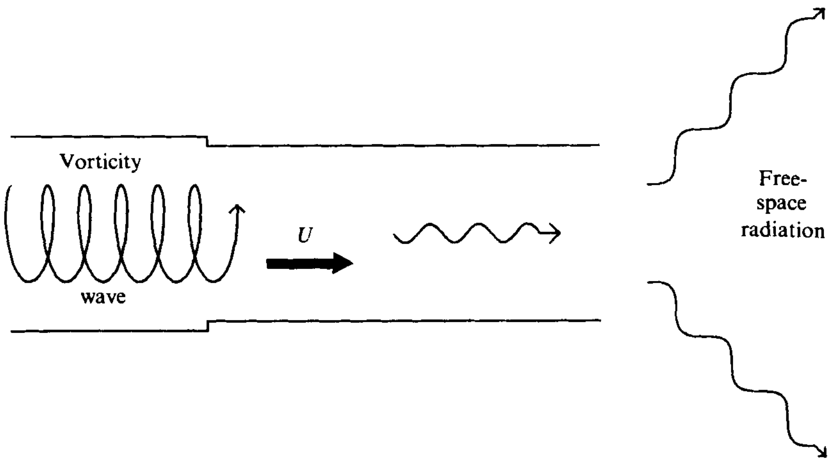

FIGURE 3. The idealized geometry of the theoretical model used to estimate the free-space radiation arising from the interaction of a vorticity wave with a contraction upstream of a nozzle exit.

the fast and slow vorticity waves of (4.6). As L/a tends to zero the dominant frequency ultimately becomes independent of L/a and is given by

$$St_m = \pi^{-1}\{\gamma a \pm 2^{\frac{1}{2}}(\Omega a/U)\}. \tag{5.1}$$

These conclusions are strictly valid only when the scattered sound is confined within an infinitely long duct. The practical problem of swirling flow in a jet pipe of finite length, and in which significant variations is duct geometry occur close to the nozzle exit, cannot be treated by the simple direct approach described in this paper. However, the above predictions may be used to estimate the magnitude of the free-space radiation in the case in which most of the sound is generated sufficiently far upstream to justify the decoupling of the scattering problem and that describing the subsequent interaction of the sound with the nozzle exit. Equations (4.15) and (4.17) then provide formulae determining plane sound waves incident on the nozzle exit from within. It has already been assumed that the characteristic acoustic wavelength is large compared with the diameter of the duct, and this implies that most of the wave energy will be reflected back into the duct at the nozzle. The amplitude of the sound which escapes into free space is of order $\omega a/c$ smaller than that of the incident wave.

The model duct configuration shown in figure 3 is probably the most appropriate in the present context, and the corresponding expression for the incident sound wave is furnished by (4.15). The free-space radiation P at a distance r from the nozzle exit has the approximate form

$$P = -\frac{i}{2}\left(\frac{a}{r}\right)\left(\frac{\omega a}{c}\right)p\left(t-\frac{r}{c}\right), \tag{5.2}$$

where $p(t)$ is the pressure perturbation of the incident sound wave at the nozzle exit (Rayleigh 1945, chap. 16). This result neglects the scattering of the sound by the exterior jet flow. In considering the problem of sound generation by entropy inhomogeneities, Ffowcs Williams & Howe (1975) have shown that (5.2) provides an

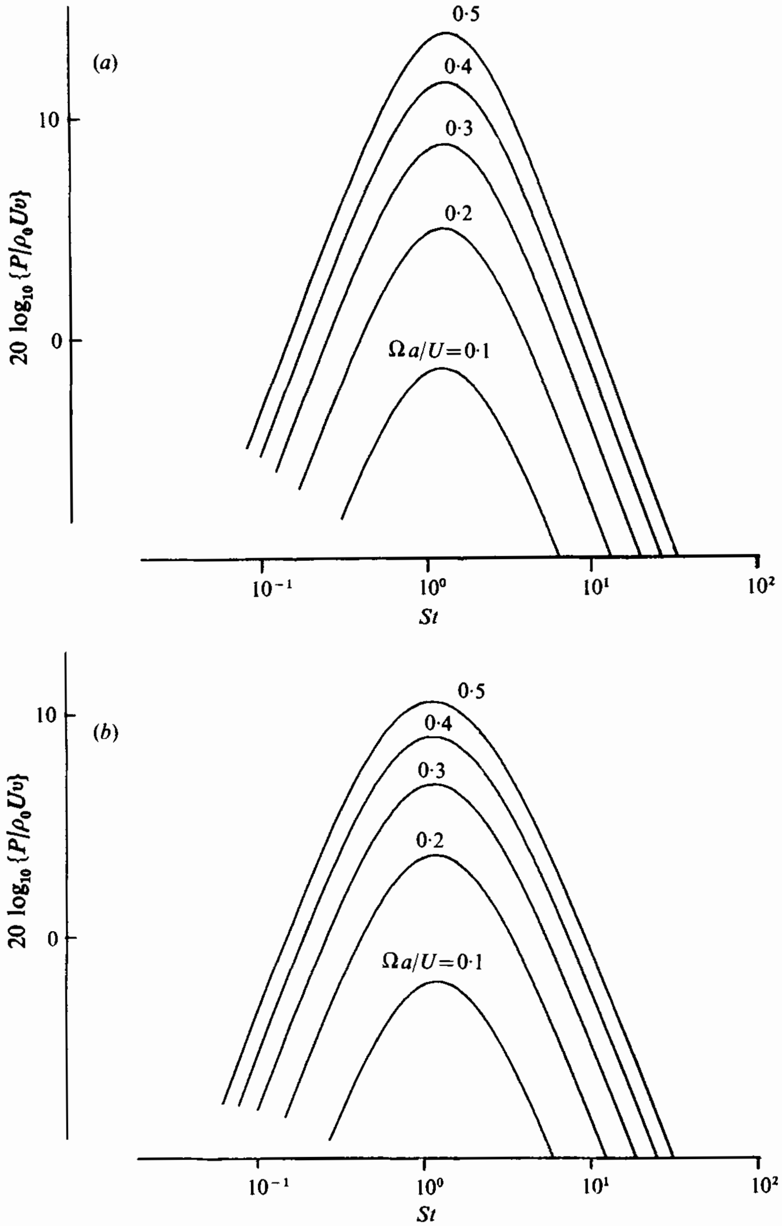

FIGURE 4. Variation of the free-space sound pressure level generated by (*a*) a 'fast' and (*b*) a 'slow' vorticity wave propagating through the contraction in the nozzle of figure 3 as a function of the Strouhal number $St = 2fa/U$ for various values of the swirl $\Omega a/U$.

adequate representation of the acoustic field even when the sound is generated well within an acoustic wavelength of the nozzle exit.

Figures 4(*a*) and (*b*) illustrate the variation of the free-space sound pressure level in dB with the Strouhal number $2fa/U$ for the fast and slow vorticity waves respectively. In each case only the lowest-order radial mode is considered, and the curves are plotted for values of the swirl $\Omega a/U$ ranging from 0·1 to 0·5, with the mean-flow velocity U

and the vorticity wave's root-mean-square velocity v held fixed. The origin of the vertical scale is undefined to within an arbitrary constant, but is the same in both figures. In all cases the peak radiation occurs at a Strouhal number of about 1–1·6 and increases steadily with the degree of swirl. Comparison of corresponding curves in the two figures shows that, for the present range of the swirl $\Omega a/U$ and at the same frequency, a fast vorticity wave is typically between 1 and 5 dB noisier than a slow wave. The precise value of the peak Strouhal number may be determined from (4.15) and (5.2). Noting that in practice $\Omega a/U$ is sufficiently small that $(\Omega a/U)^2$ may be neglected in comparison with unity and that $(\gamma a)^2$ has a minimum value of $(3{\cdot}83)^2 \gg 1$, it follows easily that the peak Strouhal number is given in the leading approximation by (5.1). In that result the plus/minus sign refers to the fast/slow vorticity waves, and therefore St_m increases with the swirl $\Omega a/U$ for the fast waves and decreases for the slow waves. For the lowest-order radial mode $\gamma a \sim 3{\cdot}83$, the peak Strouhal number is approximately 1·22, a prediction which is near to the peak Strouhal number of the pure jet mixing noise of a cold subsonic jet at angles to the jet axis close to 90° (Lush 1971). This indicates that the efficiency of a noise reduction mechanism which relies on the presence of swirl to reduce the intensity of the free-jet turbulence could well be significantly impaired by the swirl-associated excess noise source discussed in this paper.

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